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SIMILARITY SOLUTION FOR CONJUGATE NATURAL CONVECTION HEAT TRANSFER FROM A LONG VERTICAL PLATE FIN

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NOMENCLATURE

h	local heat transfer coefficient
k	fluid thermal conductivity
k_F	fin thermal conductivity
L	length of fin along base
Nu	local Nusselt number, hX_b/k
Pr	Prandtl number, ν/α
q	dimensionless local heat flux, $Nu\phi_F$
q_T	dimensionless total heat transfer per unit length of base, $Q_T/Lk(T_b-T_\infty)$
Ra	Rayleigh number, $g\beta X_b^3(T_b-T_\infty)/\nu\alpha$
t	fin half thickness
u	dimensionless vertical velocity, UX_b/α
v	dimensionless horizontal velocity, VX_b/α
X_b	location of fin base, equation (16)
x	dimensionless vertical coordinate, X/X_b
y	dimensionless horizontal coordinate, Y/X_b

Greek symbols

β	thermal coefficient of volumetric expansion
ϕ	dimensionless temperature, $(T-T_\infty)/(T_b-T_\infty)$

Subscripts

b	fin base
F	fin
∞	bulk fluid

INTRODUCTION

SIMILARITY solutions to the laminar boundary layer equations for steady natural convection from an isothermal vertical flat plate have been known since the pioneering work of Pohlhausen [1]. Sparrow and Gregg [2] added solutions for nonisothermal vertical plates with surface temperatures of the form $T-T_\infty = Ax^n$ and $B\,e^{mx}$ where x is the distance measured from the leading edge. However, Lock and Gunn [3] have shown that these two forms result in equivalent transformed equations and therefore are not distinctly separate. Yang [4] determined that the plate temperatures prescribed above cover all possible similarity solutions for steady laminar natural convection on vertical flat plates.

Natural convection heat transfer from short vertical fins of high thermal conductance can be solved by considering the fins to be isothermal and decoupling the conduction in the fin from the convection in the fluid, as in forced convection fin analysis [5, 6]. However, long fins of moderate to low conductance will not remain isothermal; this requires that the conjugate problem of conduction within the fin be solved simultaneously with natural convection in the fluid. Lock and Gunn [3] developed a similarity solution for a short tapered fin in a fluid of infinite Prandtl number. A numerical solution for conjugate heat transfer from a short vertical plate fin has been published by Sparrow and Acharya [7] at $Pr = 0.72$.

The present work outlines a similarity solution for conjugate natural convection heat transfer from a vertical fin of infinite length. Complete results are presented for a uniform conductivity plate fin as a function of the fluid Prandtl number, the only independent governing parameter.

ANALYSIS

Consider the infinitely long vertical fin shown in Fig. 1. The fin base can be selected arbitrarily provided the corresponding temperature, T_b, is known. The flow is assumed to be laminar in the Boussinesq fluid of infinite extent surrounding the fin. The fin is at a higher temperature than the surrounding isothermal bulk fluid, so the buoyant force and resulting flow is upward, toward the fin base. The problem is unchanged if the fin is cooler than the bulk fluid and the fin is inverted.

The governing conservation equations of mass, momentum and energy for the fluid can be written using boundary layer approximations as follows:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y} = 0, \tag{1}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y} = -g\beta(T-T_\infty)+\nu\frac{\partial^2 U}{\partial Y^2}, \tag{2}$$

$$U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial Y} = \alpha\frac{\partial^2 T}{\partial Y^2}. \tag{3}$$

The governing equation for a fin of variable conductivity or thickness becomes

$$\frac{\partial}{\partial X}\left(k_F t\frac{\partial T_F}{\partial X}\right)-h(T_F-T_\infty) = 0, \tag{4}$$

using the thin fin approximation. The boundary conditions on the fluid are

$$U = V = 0, \quad T = T_F \quad \text{at} \quad Y = 0, \tag{5}$$

$$U = 0, \quad T = T_\infty \quad \text{at} \quad Y = \infty, \tag{6}$$

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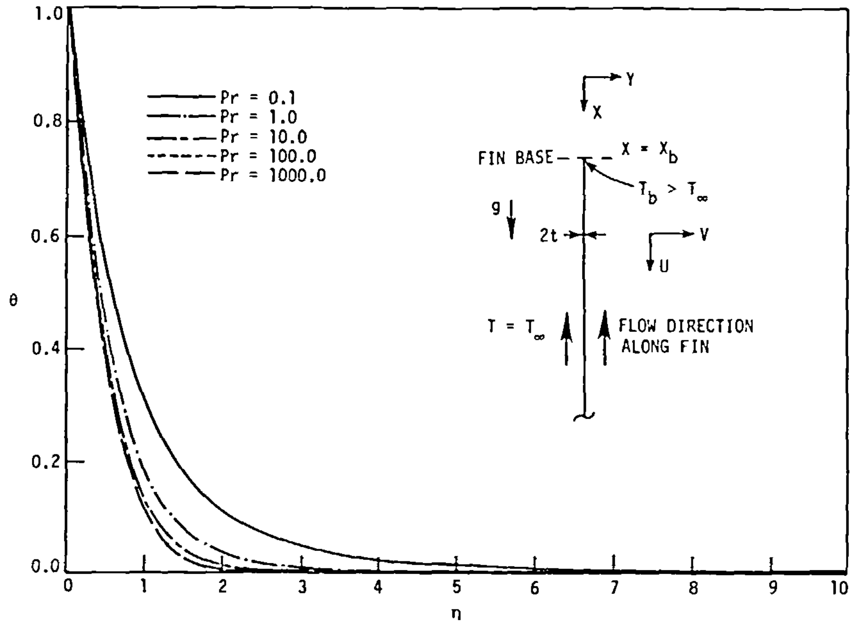

FIG. 1. Schematic diagram of an infinitely long heated vertical fin; fluid temperature profiles normal to a plate fin as a function of Prandtl number.

and the boundary conditions on the fin are

$$T_F = T_b \quad \text{at} \quad X = X_b, \tag{7}$$

$$T_F = T_\infty \quad \text{at} \quad X = \infty. \tag{8}$$

These equations can be put into dimensionless form using X_b as the length parameter. This is an arbitrary choice of length scale indicated by Yang [4] which will be determined later. Introducing a dimensionless stream function, Ψ, satisfies the continuity equation.

Considering the work of Yang [4] and Lock and Gunn [3], the only possible means of obtaining a similarity solution in the fluid adjacent to a nonisothermal fin is when the fin temperature has the form $\phi_F = x^n$. Therefore, this form is chosen for the present problem. The dimensionless similarity variables, which closely resemble those of Sparrow and Gregg [2], become

$$\eta = Cyx^{(n-1)/4}, \quad C = \frac{Ra^{1/4}}{\sqrt{2}}, \tag{9}$$

$$F(\eta) = \frac{\Psi}{4Cx^{(n+3)/4}}, \quad \theta(\eta) = \frac{\phi}{\phi_F}. \tag{10}$$

Substitution into the dimensionless fluid equations yields the transformed similarity equations

$$F''' + Pr^{-1}[(n+3)FF'' - (2n+2)(F')^2] - \theta = 0, \tag{11}$$

$$\theta'' + (n+3)F\theta' - 4nF'\theta = 0, \tag{12}$$

with the transformed boundary conditions

$$F = F' = 0, \quad \theta = 1 \quad \text{at} \quad \eta = 0, \tag{13}$$

$$F' = 0, \quad \theta = 0 \quad \text{at} \quad \eta = \infty. \tag{14}$$

The primed quantities represent derivatives with respect to η.

The fin conduction equation (4) must be analyzed to determine the matching conditions at the fin–fluid interface. Substituting the similarity transformations given in equations (9) and (10), the relation $\phi_F = x^n$, and the prescribed conductivity-thickness product, $k_F t = k_{F,b}t_b x^m$, into the fin conduction equation results in

$$\frac{k_{F,b}t_b}{X_b} x^{n+m-2}[nm + n(n-1)] + kC\theta'(0)x^{(5n-1)/4} = 0. \tag{15}$$

The powers of x must be identical so that the equation is independent of position along the fin. This prescribes the value for n, $n = 4m-7$, which is identical to the result of Lock and Gunn [3]. The indicial equation must also be satisfied, which results in the following expression for X_b:

$$X_b = \left[\frac{(4m-7)(5m-8)k_{F,b}t_b/k}{\{[g\beta(T_b - T_\infty)]/4\nu\alpha\}^{1/4}[-\theta'(0)]}\right]^{4/7}. \tag{16}$$

The similarity equations now become

$$F''' + Pr^{-1}[(4m-4)FF'' - (8m-12)(F')^2] - \theta = 0, \tag{17}$$

$$\theta'' + (4m-4)F\theta' - (16m-28)F'\theta = 0, \tag{18}$$

with the boundary conditions given in equations (13) and (14). Note that the only remaining parameters in the similarity differential equations are the fin conduction exponent m and the fluid Prandtl number.

RESULTS

The most practical value of m is zero, which corresponds to a constant property plate fin. This gives a single value for n, $n = -7$, which prescribes the plate fin temperature distribution. The complete solution for a plate fin is given below.

The similarity equations (17) and (18) with $m = 0$ subject to the boundary conditions listed in equations (13) and (14) have been solved over the range $0.001 \leqslant Pr \leqslant \infty$ using an under-overrelaxation, central difference–finite difference method. Representative plots of dimensionless temperature, stream function and vertical velocity distributions are given in Figs. 1–3.

Values of $-\theta'(0)$ were obtained by extrapolating the numerical solutions at two values of $\Delta\eta$ to zero grid size using a second order extrapolation technique [8]. The computed

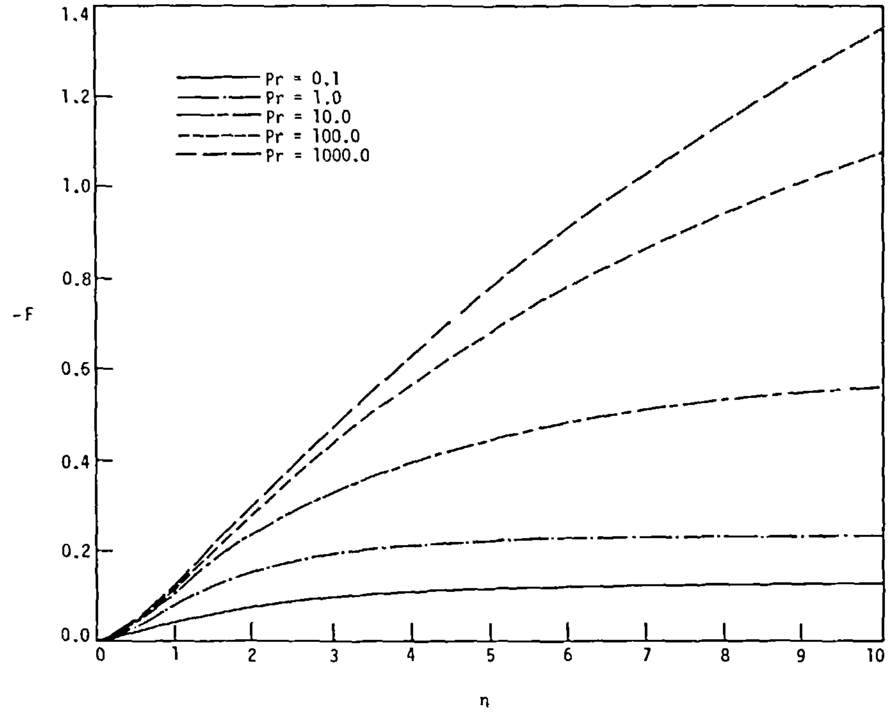

FIG. 2. Dimensionless similarity stream function for a plate fin as a function of Prandtl number.

values can be predicted to within 0.5% using the following correlation valid for $0.001 \leqslant Pr \leqslant \infty$:

$$-\theta'(0) \approx [(2.55\,Pr^{1/4})^{-2.1}+(1.539)^{-2.1}]^{-1/2.1}. \quad (19)$$

The complete solution for the plate fin ($m = 0$, $n = -7$) given in terms of dimensionless parameters is outlined below.

Length scale: X_b determined from equation (16).
Similarity variable:

$$\eta = \frac{Ra^{1/4}}{\sqrt{2}}\, y x^{-2}. \quad (20)$$

Fin temperature distribution:

$$\phi_F = x^{-7}. \quad (21)$$

Fluid temperature distribution:

$$\phi = \phi_F \theta. \quad (22)$$

Fluid vertical velocity distribution:

$$u = 2\,Ra^{1/2}F'x^{-3} = \frac{4\eta^2 x}{y^2}F'. \quad (23)$$

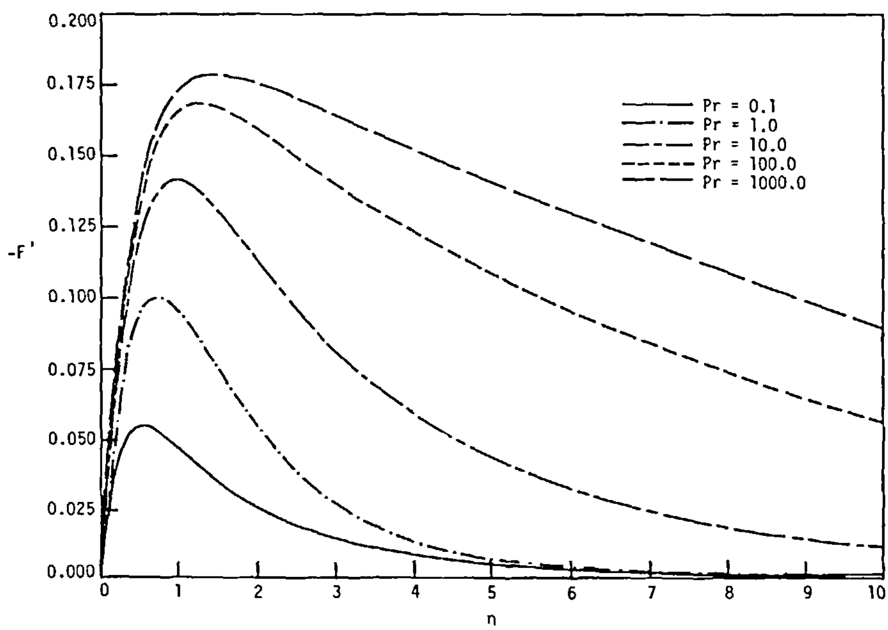

FIG. 3. Dimensionless vertical velocity profiles adjacent to a plate fin as a function of Prandtl number.

Fluid horizontal velocity distribution:

$$v = \frac{-Ra^{1/4}}{\sqrt{2}} x^{-2}\left[-4F - 8\frac{Ra^{1/4}}{\sqrt{2}} yF'x^{-2}\right]$$

$$= -\frac{\eta}{y}[-4F - 8\eta F']. \quad (24)$$

Fin local Nusselt number:

$$Nu = \frac{Ra^{1/4}}{\sqrt{2}}[-\theta'(0)]x^{-2} = \frac{\eta}{y}[-\theta'(0)]. \quad (25)$$

Fin local heat flux:

$$q = \frac{Ra^{1/4}}{\sqrt{2}}[-\theta'(0)]x^{-9} = \frac{\eta\phi_F}{y}[-\theta'(0)]. \quad (26)$$

Fin total heat transfer:

$$q_T = \frac{14k_F t}{kX_b}. \quad (27)$$

In the present solution it should be noted that the fluid approaches the fin base at $x = 1$ rather than moving in the opposite direction, away from a leading edge at $x = 0$, as it does in most other similarity solutions. The results are valid for long fins, and give a first approximation for heat transfer and fluid flow for finite length fins providing the tip temperature is nearly equal to the bulk fluid temperature and the heat transfer near the tip is an insignificant fraction of the total fin heat transfer. No direct comparison can be made between the present solution and the isothermal vertical flat plate similarity solution because of the lack of a leading edge and the breakdown of the fin conduction equation under isothermal fin conditions.

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AN APPROXIMATE SOLUTION PROCEDURE FOR LAMINAR FREE AND FORCED CONVECTION HEAT TRANSFER PROBLEMS

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NOMENCLATURE

$A,\ldots,H$	boundary layer shape factors
C_{fx}	local friction coefficient
f	velocity profile
g_x	streamwise component of gravity
Grx	local Grashof number
I, I_t	functions associated with the deviation from unity
m	parameter for the external velocity variation
m_t	parameter for the ambient temperature variation
n	parameter for the wall-ambient temperature difference variation
Nux	local Nusselt number
Pr	Prandtl number
Rex	local Reynolds number
T	temperature
ΔT_w	wall-ambient temperature difference
u	velocity in x direction
x, y	boundary layer coordinates

Greek symbols

β	coefficient of thermal expansion
δ, δ_t	viscous and thermal boundary layer thicknesses
ζ	boundary layer thickness ratio
η, η_t	similarity variables in y direction
θ	temperature profile